

MORI GEOMETRY MEETS CARTAN GEOMETRY: VARIETIES OF MINIMAL RATIONAL TANGENTS

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ABSTRACT. We give an introduction to the theory of varieties of minimal rational tangents, emphasizing its aspect as a fusion of algebraic geometry and differential geometry, more specifically, a fusion of Mori geometry of minimal rational curves and Cartan geometry of cone structures.

1. INTRODUCTION: A BRIEF PREHISTORY

Lines have been champion figures in classical geometry. Together with circles, they dominate the entire geometric contents of Euclid. Their dominance is no less strong in projective geometry. Classical projective geometry is full of fascinating results about intricate combinations of lines. As geometry entered the modern era, lines evolved into objects of greater flexibility and generality while retaining all the beauty and brilliance of classical lines. As Euclidean geometry developed into Riemannian geometry, for example, lines were replaced by geodesics which then inherited all the glory of Euclidean lines.

In the transition from classical projective geometry to complex projective geometry, real lines have been replaced by complex lines. Lines over complex numbers have all the power of lines in classical projective geometry and even more: results of greater elegance and harmony are obtained over complex numbers. A large number of results on lines and their interactions with other varieties have been obtained in complex projective geometry, their dazzling beauty no less impressive than that of classical geometry. But as complex projective geometry develops further into complex geometry and abstract algebraic geometry, which emphasize intrinsic properties of complex manifolds and abstract varieties, the notion of lines in projective space seems to be too limited for it to keep its leading role.

Firstly, to be useful in intrinsic geometry of projective varieties in projective space, lines should lie on the projective varieties. But most projective varieties do not contain lines. Even when a projective manifold contains lines, the locus of lines is, often, small and then such a locus is usually regarded as an exceptional part. Of course, there are many important varieties that are covered by lines, but they belong to a limited class from the general perspective of classification theory of varieties. In short,

() the class of projective manifolds covered by lines seems to be too special from the perspective of the general theory of complex manifolds or algebraic varieties.*

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Secondly, many of the methods employed to use lines on varieties in projective space depend on the extrinsic geometry of ambient projective space. They do not truly belong to intrinsic geometry of the varieties. Such geometric arguments are undoubtedly useful in fathoming deeper geometric properties of varieties which are described explicitly, at least to some extent. But can such methods yield results on *a priori* unknown varieties, defined abstractly by intrinsic conditions? In short,

(**) *tools employed in line geometry are not intrinsic enough to handle intrinsic problems on abstractly described varieties.*

These concerns show that lines in projective space have a rather limited role in the modern development of complex algebraic geometry. Is there a more general and more powerful notion in complex algebraic geometry that can replace the role of lines, as geodesics do in Riemannian geometry? No serious candidate had emerged until Mori's groundbreaking work [37].

In the celebrated paper [37], Mori shows that a large class of projective manifolds, including all Fano manifolds, are covered by certain intrinsically defined rational curves that behave like lines in many respects. Let us call these rational curves 'minimal rational curves'. If a projective manifold embedded in projective space is covered by lines, these lines are minimal rational curves of the projective manifold, so the notion of minimal rational curves can be viewed as an intrinsic generalization of lines.

The class of projective manifolds covered by minimal rational curves are called uniruled projective manifolds. Generalizing Mori's result, Miyaoka and Mori have proved in [32] that a projective manifold is uniruled if its anti-canonical bundle satisfies a certain positivity condition. This implies that uniruled projective manifolds form a large class of algebraic varieties. Furthermore, the minimal model program, a modern structure theory of higher-dimensional algebraic varieties, predicts that uniruled projective manifolds are precisely those projective manifolds that do not admit minimal models. Thus

projective manifolds covered by minimal rational curves form a distinguished class of manifolds, worthy of independent study from the view-point of classification theory of general projective varieties, and at the same time, large enough to contain examples of great diversity.

This overcomes the limitation (*) of the class of projective manifolds covered by lines.

Furthermore, Mori's work exhibits how to use minimal rational curves in an intrinsic way to obtain geometric information on uniruled projective manifolds. The main tool here is the deformation theory of curves, a machinery of modern complex algebraic geometry—somewhat reminiscent of the use of variational calculus in the local study of geodesics in Riemannian geometry. An example is the property that a minimal rational curve cannot be deformed when two distinct points on the curve are fixed. This result generalizes the fundamental postulate of classical geometry that "two points determine one line". The important point is that such a classical property of lines can be recovered by modern deformation theory in an abstract setting.

Deformation theory of rational curves is a powerful technique applicable to conceptual problems on varieties defined in abstract intrinsic terms.

In [37], Mori, in fact, has resolved one of the toughest problems of this kind, the Hartshorne conjecture, characterizing projective space by the positivity of the tangent bundle. The methods employed in the theory of minimal rational curves are certainly free of the concern (**) on the tools of line geometry.

These considerations indicate that minimal rational curves can serve as the natural generalization of lines, overcoming the limitations of lines, while inheriting their powerful and elegant features.

Our main interest is the geometry of minimal rational curves in uniruled projective manifolds. As in Mori's work, we would like to see how minimal rational curves can be used to control the intrinsic geometry of uniruled projective manifolds. One guiding problem is the following question on recognizing a given uniruled projective manifolds by minimal rational curves.

Problem 1.1. Let S be a (well-known) uniruled projective manifold. Given another uniruled projective manifold X , what properties of minimal rational curves on X guarantee that X is biregular (i.e. isomorphic as abstract algebraic varieties) to S ?

Here, the setting of the problem is algebraic geometry and by properties of minimal rational curves, we mean algebro-geometric properties. When S is projective space, a version of this problem is precisely what Mori solved in [37]. The initial goal of [37] was to prove the Hartshorne conjecture, which characterizes projective space by certain positivity property of the tangent bundle. After showing that the projective manifold in question is uniruled, Mori used the tangent directions of minimal rational curves to finish the proof. This part of Mori's proof has been greatly strengthened by the later work of Cho-Miyaoka-Shepherd-Barron [4], which says roughly the following (see Theorem 3.16 for a precise statement).

Theorem 1.2. *Suppose for a general point x on a uniruled projective manifold X and a general tangent direction $\alpha \in \mathbb{P}T_x(X)$, there exists a minimal rational curve through x tangent to α . Then X is projective space.*

This is a very satisfactory answer to Problem 1.1 when S is projective space. It includes, as special cases, many previously known characterizations of projective space. One may wonder why the condition on minimal rational curves here is formulated in terms of their tangent directions, not in terms of some other properties of minimal rational curves. The essential reason is because the main technical tool to handle minimal rational curves is the deformation theory of curves, as mentioned before. The tangential information of curves is essential in deformation theory. For this reason, it is natural and also useful to give conditions in terms of tangent directions of minimal rational curves.

What about other uniruled projective manifolds? When S is different from projective space, Theorem 1.2 says that minimal rational curves on S exist only in some distinguished directions. Thus in the setting of Problem 1.1, it is natural to consider

the subvariety $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ consisting of the directions of minimal rational curves through $s \in S$ and the corresponding subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$.

When S is projective space, we have $\mathcal{C}_s = \mathbb{P}T_s(S)$ for any $s \in S$. Theorem 1.2 says that a uniruled projective manifold X is projective space if and only if $\mathcal{C}_x = \mathbb{P}T_x(X)$ at some point $x \in X$. In other words, if a uniruled projective manifold has the same type of \mathcal{C}_x as projective space, then it is projective space. Based on this observation, we can refine our guiding Problem 1.1 as follows.

Problem 1.3. Let S be a (well-known) uniruled projective manifold. Given another uniruled projective manifold X , what properties of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ for general points $x \in X$ guarantee that X is biregular to S ?

Comparing this with Theorem 1.2, one may wonder why we are asking for information on \mathcal{C}_x for general points $x \in X$, instead of a single point $x \in X$ as in Theorem 1.2. This is because the information at one point $x \in X$ seems to be too weak to characterize X when $\mathcal{C}_x \neq \mathbb{P}T_x(X)$. The equality $\mathcal{C}_x = \mathbb{P}T_x(X)$ implies that minimal rational curves through one point x cover the whole of X . This is why in Theorem 1.2 the information at one point is sufficient to control the whole of X . If $\mathcal{C}_x \neq \mathbb{P}T_x(X)$, minimal rational curves through one point x cover only small part of X . Besides, the subvariety $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ may change as the point $s \in S$ varies and so the expected condition is not just on \mathcal{C}_x for a single x , but on the family

$$\{\mathcal{C}_x \subset \mathbb{P}T_x(X), \text{ general } x \in X\}.$$

This is why we are asking for the data \mathcal{C}_x for all general $x \in X$.

Now as in Problem 1.1, the properties of \mathcal{C}_x that we are looking for in Problem 1.3 are algebro-geometric properties. In algebraic geometry, however, to use properties of such a family of varieties to control the whole of X , we usually need to have good information not only on general members of the family, but also on the potential degeneration of the family. Thus it may look more reasonable to require, in Problem 1.3, some additional properties on the behavior of the family \mathcal{C}_x under degeneration. But such additional conditions would diminish the true interest of Problem 1.3. This is because in the context of intrinsic geometry of uniruled manifolds, the properties of \mathcal{C}_x we are looking for should be checkable by deformation theory of curves. Deformation theory of rational curves works well at general points of nonsingular varieties, but not so at special points. Thus it is important to find conditions for \mathcal{C}_x only for general $x \in X$ in Problem 1.3. But then controlling the whole of X using the algebraic behavior of $\{\mathcal{C}_x \subset \mathbb{P}T_x(X), \text{ general } x \in X\}$ becomes a serious issue.

This was exactly the issue puzzling me when I first encountered a version of Problem 1.3 about twenty years ago. At that time, I was working on the deformation rigidity of Hermitian symmetric spaces in the setting of algebraic geometry. I refer the reader to [11] for the details on this rigidity problem. Here it suffices to say that the deformation rigidity of Hermitian symmetric spaces was a question originated from Kodaira-Spencer's work in 1950's and the question itself did not involve rational curves. I was trying to attack this

question employing Mori's approach of minimal rational curves, which naturally led to a version of Problem 1.3 when S is an irreducible Hermitian symmetric space. In the setting of this rigidity question, I could derive a certain amount of algebro-geometric information on \mathcal{C}_x for general $x \in X$, but I was unable to figure out how to proceed from there, essentially because of the above difficulty, that it is hard to control the whole of X by algebro-geometric information on \mathcal{C}_x for general $x \in X$.

There was one hope. A few years earlier, Ngaiming Mok had overcome an obstacle of a similar kind in [33]. In that work, Mok solved what is called the generalized Frankel conjecture, which asks for a characterization of Hermitian symmetric spaces among Kähler manifolds in terms of a curvature condition. The Frankel conjecture itself is the Kähler version of the Hartshorne conjecture and was settled by Siu-Yau [40] around the time Mori solved the Hartshorne conjecture. Since the method used by Siu and Yau was rather restrictive, Mok naturally took the approach of Mori and encountered a situation similar to Problem 1.3. Now in his situation, there is a Riemannian metric on X and Mok could relate \mathcal{C}_x to a suitably deformed Riemannian metric. This enables him to show that X is Hermitian symmetric space using Berger's work on Riemannian holonomy. Roughly speaking, in [33]

the difficulty in Problem 1.3 was overcome by relating \mathcal{C}_x to a Riemannian structure.

This shows that differential geometry can be a recourse for Problem 1.3 when S is a Hermitian symmetric space. Indeed, compared with tools in algebraic geometry, methods of differential geometry tend to be more effective when the available data are only at general points of a manifold. Motivated by this, I tried to imitate Mok's argument in the setting of the deformation rigidity problem. However, the nature of the deformation rigidity problem is purely algebro-geometric and it is very hard to relate it to Riemannian structures. As a matter of fact, there had already been some unsuccessful attempts in 1960's to use Riemannian structures for the deformation rigidity question.

This was precisely the problem I was agonizing over when I attended my first ICM: Zürich 1994. Having come to the congress just to have fun listening to the new developments in mathematics, I found that Mok was there as a speaker and managed to have a chat with him. When I told him about the above difficulty in applying the approach of [33] to the deformation rigidity problem, he gave me an enlightening comment: besides the Riemannian metric, there is another differential geometric structure, a certain holomorphic G-structure, which can be used to characterize a Hermitian symmetric space. His suggestion was that one might be able to construct these G-structures using the information on \mathcal{C}_x for general $x \in X$ and from this to recover Hermitian symmetric spaces. This suggestion looked promising because algebro-geometric data are closer to holomorphic structures than Riemannian structures.

Soon after the congress, I started looking into G-structures. I realized that there is a far-reaching generalization of Riemannian structures by Elie Cartan and the G-structures modeled on Hermitian symmetric spaces are special examples of Cartan's general theory

of geometric structures. To recover these G-structures, it was necessary to investigate the geometry of \mathcal{C}_x 's in depth in the setting of the deformation rigidity problem. I had subsequent communications with Mok, and we started working together on this problem. Our collaboration was successful, leading to a solution of the deformation rigidity problem in [18]. But the most exciting point in our work was not the deformation rigidity itself. As mentioned, the essential part of [18] is to construct on X the G-structures modeled on Hermitian symmetric spaces. It turns out that a crucial point of this construction lies in a study of the behavior of \mathcal{C}_x 's not just as a family of projective algebraic varieties, but as data imposed on the tangent bundle of an open subset of X . In other words, we had to treat these \mathcal{C}_x 's as if

the union of \mathcal{C}_x 's for general $x \in X$ is a differential geometric structure.

And why not? Such a family of subvarieties in $\mathbb{P}T_x(X)$ is a legitimate example of Cartan's general geometric structures! So what happened can be summarized as follows. Initially we had been trying to relate \mathcal{C}_x 's to some differential geometric structures. These differential geometric structures were Riemannian structures in [33] and then G-structures in [18]. But actually, they have been there all along, namely, \mathcal{C}_x 's themselves!

Now once we accept \mathcal{C}_x 's as a differential geometric structure, there is no need to restrict ourselves to Hermitian symmetric spaces. This geometric structure exists for any uniruled projective manifold S and its minimal rational curves! This was an epiphany for me. We realized that the variety \mathcal{C}_x deserves a name of its own and endowed it with the appellation, somewhat uncharming, 'variety of minimal rational tangents'.

Realizing the varieties of minimal rational tangents as geometric structures opens up an approach to Problem 1.3 via Cartan geometry. In fact, Mok and I were able to show in [20] (see Theorem 4.6 for a precise statement)

Theorem 1.4. *Assume that S is a fixed uniruled projective manifold with $b_2(S) = 1$ and \mathcal{C}_s at a general point $s \in S$ is a smooth irreducible variety of positive dimension. If X is a uniruled projective manifold with $b_2(X) = 1$ and the differential geometric structures defined by \mathcal{C}_s 's and \mathcal{C}_x 's are locally equivalent in the sense of Cartan, then S and X are biregular.*

This means that for a large and interesting class of uniruled projective manifolds, Problem 1.3 can be solved by studying the Cartan geometry of the structures defined by \mathcal{C}_x 's. By Theorem 1.4, the essence of Problem 1.3 has become

searching for algebro-geometric properties of varieties of minimal rational tangents which make it possible to control the Cartan geometry of the geometric structures defined by them.

As we will see in Section 4, this search has been successful in a number of cases and Problem 1.3 has been answered for some uniruled projective manifolds, including irreducible Hermitian symmetric spaces.

Since [18], the theory of varieties of minimal rational tangents has seen exciting developments and has found a wide range of applications in algebraic geometry. For surveys

on these developments and applications, we refer the reader to [9], [19], [27] and [36]. The purpose of this article is to give an introduction to one special aspect of the theory, the development centered around Problem 1.3. This is a special aspect, because many results on varieties of minimal rational tangents and their applications are not directly related to it. Yet, this is the most fascinating aspect: it offers an area for a fusion of algebraic geometry and differential geometry, more specifically, a fusion of Mori geometry of minimal rational curves and Cartan geometry of cone structures. We will stick to the core of this aspect and will not go into the diverse issues arising from it. Those interested in further directions of explorations may find my MSRI article [13] useful.

Conventions We will work over the complex numbers and all our objects are holomorphic. Open sets refer to the Euclidean topology, unless otherwise stated. All manifolds are connected. A projective manifold is a smooth irreducible projective variety. A variety is a complex analytic set which is not necessarily irreducible, but has finitely many irreducible components. A general point of a manifold or an irreducible variety means a point in a dense open subset.

2. CARTAN GEOMETRY: CONE STRUCTURES

A priori, this section is about local differential geometry and has nothing to do with rational curves. We will introduce a class of geometric structures, cone structures, and some related notions. In a simpler form, cone structures have already appeared in twistor theory (see [31]), but as they are not widely known, I will try to give a detailed introduction.

Definition 2.1. For a complex manifold M , let $\pi : \mathbb{P}T(M) \rightarrow M$ be the projectivized tangent bundle. A *smooth cone structure* on M is a closed nonsingular subvariety $\mathcal{C} \subset \mathbb{P}T(M)$ such that all components of \mathcal{C} have the same dimension and the restriction $\varpi := \pi|_{\mathcal{C}}$ is a submersion.

We may restrict our discussion to smooth cone structures. Understanding the geometry of smooth cone structures is already challenging and lots of examples of smooth cone structures remain uninvestigated. To have a satisfactory general theory, however, we need to allow certain singularity in \mathcal{C} . This necessitates the following somewhat technical definition. (Readers not familiar with singularities may skip this definition and just stick to Definition 2.1, regarding $\nu : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ as an identity map in the subsequent discussion.)

Definition 2.2. A *cone structure* on a complex manifold M is a closed subvariety $\mathcal{C} \subset \mathbb{P}T(M)$ the normalization $\nu : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ of which satisfies the following conditions.

- (1) All components of $\tilde{\mathcal{C}}$ are smooth and have the same dimension.
- (2) The composition $\varpi := \pi \circ \nu : \tilde{\mathcal{C}} \rightarrow M$ is a submersion. In particular, the relative tangent bundle $T^{\varpi} \subset T(\tilde{\mathcal{C}})$ is a vector subbundle.
- (3) There is a vector subbundle $\mathcal{T} \subset T(\tilde{\mathcal{C}})$ with $T^{\varpi} \subset \mathcal{T}$ and $\text{rank}(\mathcal{T}) = \text{rank}(T^{\varpi}) + 1$, such that for any $\alpha \in \tilde{\mathcal{C}}$ and any $v \in \mathcal{T}_{\alpha} \setminus T^{\varpi}_{\alpha}$, the nonzero vector $d\varpi_{\alpha}(v) \in T_{\varpi(\alpha)}(M)$

satisfies

$$[d\varpi_\alpha(v)] = \nu(\alpha) \text{ as elements of } \mathbb{P}T_{\varpi(\alpha)}(M).$$

The conditions (1) and (2) say that \mathcal{C} is allowed to be singular but it becomes smooth after normalization and the natural projection to M becomes a submersion. Note that on $\mathbb{P}T(M)$, we have the tautological line bundle $\xi \subset \pi^*T(M)$. The condition (3) says that the quotient line bundle \mathcal{T}/T^ϖ is naturally isomorphic to $\nu^*\xi$. Another useful interpretation of the condition (3) is in terms of the following

Definition 2.3. Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$, let $\text{Sm}(\mathcal{C}) \subset \mathcal{C}$ be the maximal dense open subset such that

$$\pi|_{\text{Sm}(\mathcal{C})} : \text{Sm}(\mathcal{C}) \rightarrow \pi(\text{Sm}(\mathcal{C}))$$

is a submersion.

Denote by $\mathcal{T}^{\mathbb{P}T(M)} \subset T(\mathbb{P}T(M))$ the inverse image of the tautological bundle $\xi \subset \pi^*T(M)$ under $d\pi : T(\mathbb{P}T(M)) \rightarrow \pi^*T(M)$. Then the condition (3) means that the vector bundle $\mathcal{T}^{\mathbb{P}T(M)} \cap T(\text{Sm}(\mathcal{C}))$ on $\text{Sm}(\mathcal{C})$, after pulling back to $\tilde{\mathcal{C}}$ by ν , extends to a vector subbundle of $T(\tilde{\mathcal{C}})$. From this interpretation of (3), it is easy to see that

Proposition 2.4. *A cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ is a smooth cone structure if and only if \mathcal{C} is normal, i.e., the normalization $\nu : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is biholomorphic.*

All three conditions (1)-(3) for cone structures are of local nature on M . This implies

Proposition 2.5. *Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ and a connected open subset $U \subset M$, the restriction*

$$\mathcal{C}|_U := \mathcal{C} \cap \mathbb{P}T(U) \subset \mathbb{P}T(U)$$

is a cone structure on the complex manifold U .

By Proposition 2.5, we can view a cone structure as a geometric structure on M . We are interested in Cartan geometry of cone structures. In particular, isomorphisms in cone structures are given by the following

Definition 2.6. A cone structures $\mathcal{C} \subset \mathbb{P}T(M)$ on a complex manifold M is *equivalent* to a cone structure $\mathcal{C}' \subset \mathbb{P}T(M')$ on a complex manifold M' if there exists a biholomorphic map $\varphi : M \rightarrow M'$ such that the projective bundle isomorphism $\mathbb{P}d\varphi : \mathbb{P}T(M) \rightarrow \mathbb{P}T(M')$ induced by the differential $d\varphi : T(M) \rightarrow T(M')$ of φ satisfies $\mathbb{P}d\varphi(\mathcal{C}) = \mathcal{C}'$.

It is convenient to have a localized version of this:

Definition 2.7. For a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ (resp. $\mathcal{C}' \subset \mathbb{P}T(M')$) and a point $x \in M$ (resp. $x' \in M'$), we say that \mathcal{C} *at x* is *equivalent* to \mathcal{C}' *at x'* if there exists a neighborhood $U \subset M$ of x and a neighborhood $U' \subset M'$ of x' such that the restriction $\mathcal{C}|_U$ is equivalent to $\mathcal{C}'|_{U'}$ as cone structures. We say that \mathcal{C} is *locally equivalent* to \mathcal{C}' if there are points $x \in M$ and $x' \in M'$ such that \mathcal{C} at x is equivalent to \mathcal{C}' at x' .

Let us give one simple example of a cone structure. Let V be a vector space and $Z \subset \mathbb{P}V$ be a projective variety all components of which have the same dimension such that the normalization \tilde{Z} is nonsingular. Via the canonical isomorphism $T(V) = V \times V$, the projectivized tangent bundle $\mathbb{P}T(V) = V \times \mathbb{P}V$ contains the subvariety $\mathcal{C} := V \times Z \subset \mathbb{P}T(V)$. This is a cone structure. Indeed the normalization $\tilde{\mathcal{C}}$ is just $V \times \tilde{Z}$ which is smooth and $\varpi : \tilde{\mathcal{C}} \rightarrow V$ is just the projection $V \times \tilde{Z} \rightarrow V$ which is a submersion, verifying the conditions (1) and (2) of Definition 2.2. The tautological line bundle of $Z \subset \mathbb{P}V$ induces a line bundle χ in $T(V \times \tilde{Z})$ via the normalization morphism $\tilde{Z} \rightarrow Z$ and the subbundle $\mathcal{T} = T\varpi + \chi$ of $T(\tilde{\mathcal{C}})$ satisfies the condition (3).

Definition 2.8. The cone structure $V \times Z \subset \mathbb{P}T(V)$ on V defined above is called the *flat cone structure with a fiber* $Z \subset \mathbb{P}V$. We will denote it by $\text{Flat}_V^Z \subset \mathbb{P}T(V)$. A cone structure on a complex manifold M is *locally flat* if it is locally equivalent to Flat_V^Z for some $Z \subset \mathbb{P}V$ with $\dim V = \dim M$.

Definition 2.9. Let $Z \subset \mathbb{P}V$ be a projective variety. A cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ is *Z-isotrivial* if for a general $x \in M$, the fiber

$$\mathcal{C}_x = \mathcal{C} \cap \mathbb{P}T_x(M) \subset \mathbb{P}T_x(M)$$

is isomorphic to $Z \subset \mathbb{P}V$ as a projective variety, i.e., a suitable linear isomorphism $T_x(M) \rightarrow V$ sends \mathcal{C}_x to Z . A cone structure is *isotrivial* if it is Z -isotrivial for some Z .

A locally flat cone structure is isotrivial. But an isotrivial cone structure needs not be locally flat. Some isotrivial smooth cone structures are very familiar objects in differential geometry. When $Z \subset \mathbb{P}V$ is a linear subspace of dimension p , a Z -isotrivial cone structure on M is just a Pfaffian system of rank $p + 1$ on M . It is locally flat if and only if the Pfaffian system is involutive, i.e., it comes from a foliation. When $Z \subset \mathbb{P}V$ is a nonsingular quadric hypersurface, a Z -isotrivial cone structure is a conformal structure on M . It is locally flat if and only if it is locally conformally flat. A natural generalization of the conformal structure is the cone structure modeled on an irreducible Hermitian symmetric space $S = G/P$. The isotropy action of P on the tangent space $T_o(S)$ at the base point $o \in S$ has a unique closed orbit $\mathcal{C}_o \subset \mathbb{P}T_o(S)$. A Z -isotrivial cone structure where $Z \subset \mathbb{P}V$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is called an *almost S-structure*. A conformal structure is exactly an almost S -structure where S is a nonsingular quadric hypersurface, equivalently, an irreducible Hermitian symmetric space of type IV. The natural almost S -structure $\mathcal{C} \subset \mathbb{P}T(S)$ given by the translate of \mathcal{C}_o by G -action is locally flat, which can be seen by Harish-Chandra coordinates of irreducible Hermitian symmetric spaces (see Section (1.2) in [35] for a presentation in terms of explicit coordinates for Grassmannians). The G -structure on an irreducible Hermitian symmetric space S referred to in Section 1 is essentially equal to the cone structure $\mathcal{C} \subset \mathbb{P}T(S)$.

How do we check the local equivalence of two cone structures? A general method of checking equivalence of geometric structures has been formulated by Elie Cartan [2]. The fundamental apparatus in Cartan's method is a coframe.

Definition 2.10. Let V be a vector space and let M be a complex manifold with $\dim V = \dim M$. A *coframe* on M is a trivialization $\omega : T(M) \rightarrow M \times V$, equivalently, a V -valued 1-form on M such that $\omega_x : T_x(M) \rightarrow V$ is an isomorphism for each $x \in M$. We will denote by $\mathbb{P}\omega : \mathbb{P}T(M) \rightarrow M \times \mathbb{P}V$ the trivialization of the projectivized tangent bundle induced by ω . Given a coframe, there exists a $\text{Hom}(\wedge^2 V, V)$ -valued function σ^ω on M , called the *structure function* of ω , such that

$$d\omega = \sigma^\omega_\bullet(\omega \wedge \omega).$$

A coframe is *closed* if $d\omega = 0$, i.e., the structure function σ^ω is identically zero. A coframe is *conformally closed* if there exists a holomorphic function f on an open subset $U \subset M$ such that $f\omega$ is closed on U .

The following is a simple consequence of the Poincaré lemma (see Theorem 3.4 in [12]).

Proposition 2.11. *Let $V^\vee \subset \text{Hom}(\wedge^2 V, V)$ be the natural inclusion of the dual space of V given by contracting with one factor. When $\dim M \geq 3$, a coframe ω is conformally closed if and only if σ^ω takes values in V^\vee .*

Although Cartan's method is applicable to the equivalence problem for arbitrary cone structures, its actual implementation can be challenging, depending on the type of the cone structure. For isotrivial cone structures, however, this becomes simple:

Definition 2.12. Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a Z -isotrivial cone structure for a projective variety $Z \subset \mathbb{P}V$. A coframe $\omega : T(M) \rightarrow M \times V$ is *adapted* to the cone structure if $\mathbb{P}\omega(\mathcal{C}) = Z$.

Proposition 2.13. *An isotrivial cone structure is locally flat if and only if after restricting to an open subset, it admits a conformally closed adapted coframe.*

Since an isotrivial cone structure always admits an adapted coframe, we can use Proposition 2.11 and Proposition 2.13 to check the local flatness of an isotrivial cone structure. One difficulty here is that there may be several different adapted coframes, so we need to choose the right one. Different choices of adapted coframes are related by the linear automorphism group of the fiber. Let us elaborate this point.

For a projective variety $Z \subset \mathbb{P}V$, let $\widehat{Z} \subset V$ be its homogeneous cone. Denote by $\text{Aut}(\widehat{Z}) \subset \text{GL}(V)$ the linear automorphism group of \widehat{Z} and by $\text{aut}(\widehat{Z}) \subset \text{gl}(V)$ its Lie algebra. Since $\widehat{Z} \subset V$ is a cone, the Lie algebra $\text{aut}(\widehat{Z})$ always contains the scalars \mathbb{C} . When $\text{aut}(\widehat{Z}) = \mathbb{C}$, a Z -isotrivial cone structure has a unique adapted coframe up to multiplication by functions. Consequently, the method of Proposition 2.13 essentially determines the local flatness of Z -isotrivial cone structures when $\text{aut}(\widehat{Z}) = \mathbb{C}$.

When $\text{aut}(\widehat{Z}) \neq \mathbb{C}$, however, compositions with $\text{Aut}(\widehat{Z})$ -valued functions give rise to many different choices of adapted coframes for a Z -isotrivial cone structure. In this case, Proposition 2.13 is not decisive and we have to consider the problem of choosing the right coframe. This leads to the equivalence problem for G-structures where G corresponds to the group $\text{Aut}(\widehat{Z}) \subset \text{GL}(V)$. The general theory of G-structures has been developed by many mathematicians. In particular, for the G-structures modeled on Hermitian symmetric

spaces, [5] and [39] provide a calculable criterion for local flatness in terms of the vanishing of certain ‘curvature tensors’, which are more elaborate version of the structure functions σ^ω .

It turns out that the cone structures we are interested in are equipped with some additional structures.

Definition 2.14. Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a cone structure. From the condition (3) in Definition 2.2, we have an exact sequence of vector bundles on $\tilde{\mathcal{C}}$

$$0 \rightarrow T^\varpi \rightarrow \mathcal{T} \rightarrow \nu^*\xi \rightarrow 0.$$

A line subbundle $\mathcal{F} \subset \mathcal{T}$ is called a *connection* of the cone structure if \mathcal{F} splits this exact sequence. Thus a connection exists if and only if this exact sequence splits.

All the cone structures we are to meet have certain canonically defined connections. These connections will have some special properties.

Definition 2.15. In Definition 2.14, \mathcal{F} is a *characteristic connection* if $[\mathcal{F}, [\mathcal{F}, \mathcal{T}]] \subset [\mathcal{F}, \mathcal{T}]$ at general points of $\tilde{\mathcal{C}}$. The inclusion means that for any local section f of \mathcal{F} and any local section v of T^ϖ regarded as local vector fields in some open subset of $\tilde{\mathcal{C}}$, the Lie bracket $[f, [f, v]]$ is a local section of $[\mathcal{F}, \mathcal{T}]$.

The most important property of a characteristic connection is its uniqueness for a large class of cone structures. This condition is formulated in terms of the Gauss map and the projective second fundamental form. Let us recall the definition.

Definition 2.16. Let $Z \subset \mathbb{P}V$ be an irreducible projective variety of dimension p . The *Gauss map* of Z is the morphism $\gamma : \text{Sm}(Z) \rightarrow \text{Gr}(p+1, V)$ defined on the smooth locus of Z by associating to a smooth point α of Z the affine tangent space $T_\alpha(\hat{Z}) \subset V$, the tangent space of the homogeneous cone $\hat{Z} \subset V$ along α . We say that the Gauss map of Z is *nondegenerate* if γ is generically finite over its image. Let $\alpha \in \text{Sm}(Z)$ be a smooth point of Z and let $N_{Z,\alpha}$ be the normal space of Z inside $\mathbb{P}V$ at α . The differential of γ defines a homomorphism

$$\text{II}_{Z,\alpha} : \text{Sym}^2 T_\alpha(Z) \rightarrow N_{Z,\alpha},$$

called the *projective second fundamental form*. We say that $\text{II}_{Z,\alpha}$ is *nondegenerate* if its null space

$$\text{Null}_{\text{II}_{Z,\alpha}} = \{v \in T_\alpha(Z), \text{II}_{N_{Z,\alpha}}(v, u) = 0 \text{ for all } u \in T_\alpha(Z)\}$$

is zero. Then the Gauss map of Z is nondegenerate if the projective second fundamental form of Z is nondegenerate at a general point of Z . It is well-known (e.g. Theorem 3.4.2 in [25]) that if an irreducible projective variety Z is smooth and not a linear subspace of $\mathbb{P}V$, then the Gauss map of Z is nondegenerate, or equivalently, the projective second fundamental form $\text{II}_{Z,\alpha}$ is nondegenerate at a general point $\alpha \in Z$.

The uniqueness result for a characteristic connection is the following result from [21].

Theorem 2.17. *Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a cone structure such that all components of the fiber \mathcal{C}_x for a general $x \in M$ have nondegenerate Gauss maps. Then \mathcal{C} has at most one characteristic connection.*

It is easy to see that the flat cone structure Flat_V^Z in Definition 2.8 has a characteristic connection given by the intersection of \mathcal{T} with the fibers of the projection map $V \times \tilde{Z} \rightarrow \tilde{Z}$. Cone structures admitting characteristic connections have certain amount of flatness, although this is not easy to explicate. One manifestation is the following proposition (see Theorem 6.2 in [13] for a proof). Although it is stated here without any regard to minimal rational curves, a version of this proposition is first discovered in [18] for varieties of minimal rational tangents and has been the key revelation on the significance of the differential geometric interpretation of the varieties of minimal rational tangents, as mentioned in Section 1.

Theorem 2.18. *Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ admitting a characteristic connection, denote by $\text{Pf}(\mathcal{C})$ the Pfaffian system defined on a dense open subset of M by the linear span of the homogeneous cone $\hat{\mathcal{C}} \subset T(M)$. Then for any $\alpha \in \text{Sm}(\hat{\mathcal{C}})$ and $\beta \in T_\alpha(\hat{\mathcal{C}}) \cap T_\alpha^\pi$, and any local sections $\vec{\alpha}$ and $\vec{\beta}$ of $\text{Pf}(\mathcal{C})$ extending α and β , the Lie bracket $[\vec{\alpha}, \vec{\beta}]$ belongs to $\text{Pf}(\mathcal{C})$ at the point $\pi(\alpha)$.*

When $\mathcal{C} \subset \mathbb{P}T(M)$ itself is a Pfaffian system, i.e., when $\text{Pf}(\mathcal{C}) = \mathcal{C}$, Proposition 2.18 says that the existence of a characteristic connection on \mathcal{C} implies that \mathcal{C} is involutive. This is an example of the statement that characteristic connections contain certain amount of flatness. Another example of this phenomena is the next result from [12]:

Theorem 2.19. *Let $Z \subset \mathbb{P}V$ be a smooth hypersurface of degree ≥ 4 . Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a Z -isotrivial cone structure. If \mathcal{C} has a characteristic connection, then it is locally flat.*

The above results show that the existence of characteristic connections impose severe restrictions on isotrivial cone structures. We expect similar restrictions on non-isotrivial cone structures, although no specific results are known.

Being a characteristic connection is a local property of a connection \mathcal{F} on $\tilde{\mathcal{C}}$: the condition in Definition 2.15 is to be checked on an open subset of $\tilde{\mathcal{C}}$. The connections we are interested in have another important property which is of a global nature. To introduce this property, we note that there is a natural vector subbundle $\mathcal{P} \subset T(\text{Sm}(\mathcal{C}))$ defined as follows. At a smooth point $\alpha \in \mathcal{C}$ with $x = \pi(\alpha)$, we have the differential $d\varpi_\alpha : T_\alpha(\mathcal{C}) \rightarrow T_x(M)$ of the projection $\varpi = \pi|_{\text{Sm}(\mathcal{C})} : \text{Sm}(\mathcal{C}) \rightarrow M$. Since the fiber $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ of $\pi|_{\mathcal{C}}$ is smooth at α by Definition 2.2 (2), we have the affine tangent space $T_\alpha(\hat{\mathcal{C}}_x) \subset T_x(M)$. Define $\mathcal{P}_\alpha \subset T_\alpha(\mathcal{C})$ by

$$\mathcal{P}_\alpha := d\varpi_\alpha^{-1}(T_\alpha(\hat{\mathcal{C}}_x)).$$

This defines a vector bundle \mathcal{P} on $\text{Sm}(\mathcal{C})$.

Definition 2.20. View $T^\varpi \subset \mathcal{T}$ on $\tilde{\mathcal{C}}$ and a connection $\mathcal{F} \subset \mathcal{T}$ as vector bundles on $\text{Sm}(\mathcal{C})$ via the normalization morphism $\nu : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ which is an isomorphism over $\text{Sm}(\mathcal{C})$. Then

we have $\mathcal{T} \subset \mathcal{P} \subset T(\text{Sm}(\mathcal{C}))$. A connection $\mathcal{F} \subset \mathcal{T}$ is \mathcal{P} -*splitting* if there exists a vector subbundle $\mathcal{W} \subset \mathcal{P}$ on $\text{Sm}(\mathcal{C})$ that splits

$$0 \rightarrow T^\varpi \rightarrow \mathcal{P} \rightarrow \mathcal{P}/T^\varpi \rightarrow 0$$

such that $\mathcal{W} \cap \mathcal{T} = \mathcal{F}$ on $\text{Sm}(\mathcal{C})$.

The connections we are interested in are \mathcal{P} -splitting. The significance of the \mathcal{P} -splitting property has been noticed only recently in [14], and many of its implications are yet to be discovered. It is used in [14] in the following way.

Theorem 2.21. *Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a smooth cone structure of codimension 1. In other words, \mathcal{C} is a smooth hypersurface in $\mathbb{P}T(M)$ such that $\varpi : \mathcal{C} \rightarrow M$ is a submersion. If \mathcal{C} has a \mathcal{P} -splitting connection and $\dim M \geq 4$, then it is isotrivial.*

In particular, if the degree of the fiber \mathcal{C}_x in Theorem 2.21 is at least 4 and the \mathcal{P} -splitting connection is also a characteristic connection, then \mathcal{C} is locally flat by Theorem 2.19. Actually, the requirement in Theorem 2.19 that the degree d of the hypersurface is at least 4 can be weakened to $d \geq 3$ if the connection is \mathcal{P} -splitting. Thus at least for smooth cone structures of codimension 1, the existence of a \mathcal{P} -splitting connection has a significant consequence. This is to be contrasted with cone structures of codimension 1 that are not smooth. There are examples discovered in [3] of cone structures of codimension 1 that have \mathcal{P} -splitting characteristic connections but are not isotrivial.

Cartan geometry of cone structures with \mathcal{P} -splitting characteristic connections is our central interest from the differential geometric side. As we will see in Section 4, there are lots of examples of cone structures with \mathcal{P} -splitting characteristic connections. Properties of such structures are intricately related to the projective geometry of the fibers \mathcal{C}_x . Thus this Cartan geometry has an inseparable link with projective algebraic geometry. This is analogous to the fact that Cartan geometry of G-structures has an intimate link with representation theory of Lie groups. For example, the proofs of Theorem 2.19 and Theorem 2.21 use cohomological properties of smooth hypersurfaces in projective space. The number of results in this direction is still very small and the investigation of cone structures with \mathcal{P} -splitting characteristic connections is a wide open area.

Let us close this section with one remark. There is an additional property, which could be called the *admissibility* of a connection, that holds for all connections we are interested in. This property arises from Bernstein-Gindikin's admissibility condition in integral geometry [1]. Significant consequences of admissibility have not yet been found in connection with the topic of this article, which is the reason I have skipped discussing this property. However, this additional condition may lead to interesting discoveries in the future.

3. MORI GEOMETRY: MINIMAL RATIONAL CURVES

Our major interest in algebraic geometry is in uniruled projective manifolds, i.e., projective manifolds covered by rational curves. Recall that a rational curve C on a projective manifold X is a curve $C \subset X$ with normalization $\nu_C : \mathbb{P}^1 \rightarrow C$ by \mathbb{P}^1 . The set

$\text{RatCurves}(X)$ of all rational curves on X can be given a scheme structure and its normalization is denoted by $\text{RatCurves}^n(X)$. Each irreducible component \mathcal{K} of $\text{RatCurves}^n(X)$ is a quasi-projective variety equipped with the universal \mathbb{P}^1 -bundle $\rho_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow \mathcal{K}$ and the associated cycle morphism $\mu_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow X$. This means that for each $z \in \mathcal{K}$, the corresponding rational curve $C \subset X$ is given by $\mu_{\mathcal{K}}(\rho_{\mathcal{K}}^{-1}(z))$ and the morphism

$$\nu_C := \mu_{\mathcal{K}}|_{\rho_{\mathcal{K}}^{-1}(z)} : \mathbb{P}^1 \rightarrow C = \mu_{\mathcal{K}}(\rho_{\mathcal{K}}^{-1}(z))$$

is the normalization of C . For a rigorous presentation of this foundational material, we refer the reader to [28].

Now I am going to introduce a number of terms related to uniruled manifolds and rational curves. I should warn the reader that most of these are *not standard*: they appear under different names in the literature. As is the case in any growing area of mathematics, the technical terms have not yet been completely standardized. I believe the terms introduced below are shorter and more intuitive than some of the ones in use (including some in my own papers) for nonexperts to remember their meaning. You may regard the definitions below as nicknames we will use in this article. To start with, we can give a precise definition of a uniruled projective manifold in the following form.

Definition 3.1. An irreducible component \mathcal{K} of $\text{RatCurves}^n(X)$ is called a *uniruling* on X if the cycle morphism $\mu_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow X$ is dominant. A projective manifold X is *uniruled* if it has a uniruling. For a line bundle L on X , we will denote by $\deg_L(\mathcal{K})$ the L -degree of a member of \mathcal{K} .

A fundamental tool in the study of unirulings on X is the deformation theory of rational curves on X , or equivalently, the deformation theory of morphisms $\mathbb{P}^1 \rightarrow X$. By the classical Kodaira theory, given a rational curve $C \subset X$, the first-order deformation of the normalization morphism $\nu_C : \mathbb{P}^1 \rightarrow X$ regarded as a map to X is controlled by the pull-back $\nu_C^*T(X)$ of the tangent bundle of X . In this regard, the following definition is fundamental.

Definition 3.2. A rational curve $C \subset X$ is *free* if $\nu_C^*T(X)$ is semi-positive, i.e., of the form $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, $n = \dim X$, with $a_i \geq 0$ for all i .

Free rational curves have a nice deformation theory because $H^1(\mathbb{P}^1, \nu_C^*T(X)) = 0$ by the semi-positivity of $\nu_C^*T(X)$. This cohomology group contains the obstruction to realizing deformations of ν_C from its infinitesimal deformations in $H^0(\mathbb{P}^1, \nu_C^*T(X))$. Thus the vanishing implies the following.

Theorem 3.3. Let \mathcal{K} be an irreducible component of $\text{RatCurves}^n(X)$. Denote by $\mathcal{K}^{\text{free}} \subset \mathcal{K}$ the parameter space of members of \mathcal{K} that are free. Then \mathcal{K} is a uniruling if and only if $\mathcal{K}^{\text{free}}$ is nonempty. In this case, $\mathcal{K}^{\text{free}}$ is a Zariski open subset of the smooth locus of \mathcal{K} .

Given a uniruling \mathcal{K} on X and a point $x \in X$, let \mathcal{K}_x be the normalization of the subvariety of \mathcal{K} parametrizing members of \mathcal{K} passing through x . When x is a general point of X , the structure of \mathcal{K}_x is particularly nice:

Theorem 3.4. *For a uniruling \mathcal{K} on a projective manifold X and a general point $x \in X$, all members of \mathcal{K}_x belongs to $\mathcal{K}^{\text{free}}$. Furthermore, the variety \mathcal{K}_x is a finite union of smooth quasi-projective varieties of dimension $\deg_{K_X^{-1}}(\mathcal{K}) - 2$.*

Both Theorem 3.3 and Theorem 3.4 must have been known before [37], although their significance has not been fully recognized until Mori's work. Now we are ready to introduce minimal rational curves.

Definition 3.5. A uniruling \mathcal{K} on a projective manifold X is *unbreakable* if \mathcal{K}_x is projective for a general $x \in X$. In other words, \mathcal{K} is an unbreakable uniruling if a general fiber of the cycle morphism $\mu_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow X$ is nonempty and complete. Members of an unbreakable uniruling on X will be called *minimal rational curves* on X .

Unbreakable unirulings exist on any uniruled projective manifold. To see this, we need the following notion.

Definition 3.6. Let L be an ample line bundle on a projective manifold X . A uniruling \mathcal{K} is a *minimal with respect to L* , if $\deg_L(\mathcal{K})$ is minimal among all unirulings of X . A uniruling is a *minimal uniruling* if it is minimal with respect to some ample line bundle. Minimal unirulings exist on any uniruled projective manifold and they are unbreakable.

It is essential to understand the geometric idea behind the unbreakability of minimal unirulings. Suppose for a uniruling \mathcal{K} , which is minimal with respect to an ample line bundle L , the variety \mathcal{K}_x is not projective for a general point $x \in X$. Then the members of \mathcal{K}_x degenerate to reducible curves all components of which are rational curves of smaller L -degree than the members of \mathcal{K} and some components of which pass through x . Collecting those components passing through x as x varies over the general points of X gives rise to another uniruling \mathcal{K}' satisfying $\deg_L(\mathcal{K}') < \deg_L(\mathcal{K})$, a contradiction to the minimality of $\deg_L(\mathcal{K})$. This argument gives an intuitive picture behind the definition of an unbreakable uniruling: if a uniruling is not unbreakable, its members can be broken into members of another uniruling. More figuratively speaking, *if a uniruling is not unbreakable, it can be broken into a smaller uniruling*.

It is worth introducing a special class of minimal unirulings, which are particularly interesting from the viewpoint of (extrinsic) projective algebraic geometry:

Definition 3.7. Let L be an ample line bundle on a projective manifold X that is base-point free. A uniruling \mathcal{K} on X is a *uniruling by lines* if $\deg_L(\mathcal{K}) = 1$. Geometrically, this means that there is a morphism $j : X \rightarrow \mathbb{P}^N$ which is finite over $j(X)$ and sends members of \mathcal{K} to lines in \mathbb{P}^N .

Most of the classical examples of unbreakable unirulings are unirulings by lines and the morphism j is often an embedding. For example, smooth complete intersections of low degree in \mathbb{P}^N are covered by lines and so are Grassmannians under the Plücker embedding. But there are many examples of unirulings by lines where j is not an embedding. Also there are many minimal unirulings which are not unirulings by lines: hypersurfaces of degree n

in \mathbb{P}^n have minimal unirulings by conics. Also there are many unbreakable unirulings that are not minimal. Trivial examples can be constructed on a product $X = X_1 \times X_2$ of two uniruled manifolds. There are more interesting examples in [3] and [30] where these non-minimal unbreakable unirulings rather than minimal unirulings play crucial roles. We refer the reader to [24] and [28] for many examples of unbreakable unirulings.

All these examples illustrate that unbreakable unirulings and minimal rational curves, which exist on any uniruled projective manifolds, are genuine extensions of the classical notion of unirulings by lines:

$$\{\text{unirulings by lines}\} \subset \{\text{minimal unirulings}\} \subset \{\text{unbreakable unirulings}\}.$$

The important point is that this extension retains many geometric properties of unirulings by lines. The most fundamental example is the following. Let $x \neq y$ be two distinct points on a projective manifold X . If \mathcal{K} is a uniruling by lines on X , we know that there exists at most one member of \mathcal{K} through x and y . Mori shows that a weaker version of this property continues to hold for unbreakable unirulings:

Theorem 3.8. *Let \mathcal{K} be an unbreakable uniruling. Then for a general point $x \in X$ and any other point $y \in X$, there does not exist a positive-dimensional family of members of \mathcal{K} that pass through both x and y . In particular, $\dim \mathcal{K}_x \leq \dim X - 1$.*

Theorem 3.8 is proved by what is called the ‘bend-and-break’ argument. Geometrically, it says that any 1-dimensional family of rational curves which share two distinct points in common must degenerate into a reducible curve. This is the most important geometric property of an unbreakable uniruling. Combined with Theorem 3.4, the bound on $\dim \mathcal{K}_x$ in Theorem 3.8 implies that $\deg_{K_X^{-1}}(\mathcal{K}) \leq \dim X + 1$ for any unbreakable uniruling. In other words, the K_X^{-1} -degrees of minimal rational curves are bounded by $\dim X + 1$. The fact that all uniruled projective manifolds are covered by rational curves of small degree is of fundamental significance and has eventually developed into the boundedness of Fano manifolds, for which we refer the reader to Chapter V of [28].

An infinitesimal version of Theorem 3.8 is important for us. A key notion here is the following

Definition 3.9. A rational curve $C \subset X$ is *unbending* if under the normalization $\nu_C : \mathbb{P}^1 \rightarrow C \subset X$, the vector bundle $\nu_C^* T(X)$ has the form

$$\nu_C^* T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

for some integer p satisfying $0 \leq p \leq n - 1$, where $n = \dim X$.

What is the rationale behind the name ‘unbending’? Note that a free rational curve $C \subset X$ is unbending if and only if for any two distinct points $x \neq y \in \mathbb{P}^1$ and their maximal ideals \mathfrak{m}_x and \mathfrak{m}_y in $\mathcal{O}_{\mathbb{P}^1}$,

$$H^0(\mathbb{P}^1, \nu_C^* T(X) \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y) = H^0(\mathbb{P}^1, T(\mathbb{P}^1) \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y).$$

In the standard deformation theory, this means that C does not have infinitesimal deformation fixing two distinct points. Figuratively, we can say that ‘an unbending curve cannot

be bent infinitesimally'. Comparing this with Theorem 3.8, we wonder whether members of an unbreakable uniruling are unbending. This is indeed the case for general members:

Theorem 3.10. *A general member of an unbreakable uniruling is unbending.*

Theorem 3.10 is not a direct consequence of Theorem 3.8 because the notion of an unbending rational curve gives information on infinitesimal deformation only, while Theorem 3.8 is concerned with an actually realized deformation. Theorem 3.10 enables us to control the behavior of the tangent directions of members of an unbreakable uniruling. To study this behavior systematically, it is convenient to introduce the tangent map.

Definition 3.11. For any uniruling \mathcal{K} on a projective manifold X and a point $x \in X$, the rational map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}T_x(X)$ sending a member of \mathcal{K}_x that is smooth at x to its tangent direction is called the *tangent map* at x .

If C is an unbending member of \mathcal{K}_x , the tangent map can be extended to $[C] \in \mathcal{K}_x$, even when C is singular at x , because the differential $d\nu_C : T(\mathbb{P}^1) \rightarrow \nu_C^*T(X)$ is injective. In fact, a stronger result holds.

Theorem 3.12. *In Definition 3.11, if C is an unbending member of \mathcal{K}_x , the tangent map τ_x is well-defined and immersive at $[C] \in \mathcal{K}_x$.*

In particular, Theorem 3.10 implies that for an unbreakable uniruling \mathcal{K} and a general point $x \in X$, the tangent map τ_x is generically finite over its image.

It is worth comparing Theorem 3.12 with Theorem 3.3. Theorem 3.3 suggests that the freeness of rational curves is an ‘individualized’ version of the notion of a uniruling. In a similar way, Theorem 3.8 suggests that the unbending property of rational curves is an ‘individualized’ version of the notion of an unbreakable uniruling. The correspondence in the latter case, however, is less exact: general members of an unbreakable uniruling are unbending, but a uniruling whose general members are unbending is not necessarily unbreakable.

There is another important difference between the two correspondences. When \mathcal{K} is a uniruling, we know that every member of \mathcal{K}_x for a general $x \in X$ is free by Theorem 3.4. Is it true that for an unbreakable uniruling \mathcal{K} , every member of \mathcal{K}_x for a general $x \in X$ is unbending? Using Theorem 3.12, we can formulate this question as follows:

Question 3.13. For an unbreakable uniruling \mathcal{K} on a projective manifold X , is the tangent map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}T_x(X)$ at a general point $x \in X$ extendable to an immersion $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$?

It is easy to see that if \mathcal{K} is a uniruling by lines, then the answer is yes: τ_x is an embedding. This corresponds to the classical property of lines that they are determined by their tangent directions at a given point. Encouraged by this special case, it has been expected that the answer to Question 3.13 is affirmative for all unbreakable unirulings, or at least for all minimal unirulings. Recently, however, counterexamples have been discovered: an unbreakable one in [3] and then a minimal one in [17].

Although not all members of \mathcal{K}_x are as nice as we would wish them to be, they are still considerably well behaved, as the following result of Kebekus shows. In [26], an in-depth analysis of singularities of members of \mathcal{K}_x has been carried out. Among other things, Kebekus has shown

Theorem 3.14. *For an unbreakable uniruling \mathcal{K} and a general point $x \in X$, all members of \mathcal{K}_x are immersed at the point corresponding to x .*

To elaborate, a member of \mathcal{K}_x is given by a morphism $\nu_C : \mathbb{P}^1 \rightarrow C \subset X$ with a point $o \in \mathbb{P}^1$ satisfying $\nu_C(o) = x$. Theorem 3.14 says that $(d\nu_C)_o : T_o(\mathbb{P}^1) \rightarrow T_x(X)$ is injective. Using this, Kebekus has shown the following important result.

Theorem 3.15. *In the setting of Theorem 3.14, the tangent morphism $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ can be defined by assigning to each member C of \mathcal{K}_x its tangent direction*

$$\mathbb{P}(d\nu_C(T_o(\mathbb{P}^1))) \in \mathbb{P}T_x(X).$$

This morphism τ_x is finite over its image.

Kebekus's study of singularities of members of \mathcal{K}_x , including Theorem 3.14, plays an important role in the proof of Theorem 1.2 due to Cho, Miyaoka and Shepherd-Barron [4]. In terms of Theorem 3.15, we can state it as follows.

Theorem 3.16. *Let X be a uniruled projective manifold and let \mathcal{K} be an unbreakable uniruling. Suppose for a general point $x \in X$, the tangent morphism $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ is dominant. Then X is projective space and \mathcal{K} is the space of lines.*

By Theorem 3.15, the morphism τ_x in Theorem 3.16 is a finite covering of projective space. The essential point in the proof of Theorem 3.16 is to prove that τ_x is a birational morphism, thus an isomorphism. In fact, it is fairly easy to show that X is projective space if τ_x is an isomorphism. In this sense, the following result in [21] is a generalization of Theorem 3.16.

Theorem 3.17. *Let X be a uniruled projective manifold and let \mathcal{K} be an unbreakable uniruling. For a general point $x \in X$, the tangent morphism $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ is birational over its image.*

Combining Theorem 3.15 and Theorem 3.17, we see that τ_x is the normalization of its image in $\mathbb{P}T_x(X)$.

The collection of tangent morphisms $\{\tau_x, \text{ general } x \in X\}$ can be assembled into a single map. Recall that we have the universal \mathbb{P}^1 -bundle $\rho_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow \mathcal{K}$ and the cycle morphism $\mu_{\mathcal{K}} : \text{Univ}_{\mathcal{K}} \rightarrow X$. When \mathcal{K} is unbreakable, the variety \mathcal{K}_x for a general $x \in X$ can be identified with $\mu_{\mathcal{K}}^{-1}(x) \subset \text{Univ}_{\mathcal{K}}$ and we have a rational map $\tau : \text{Univ}_{\mathcal{K}} \dashrightarrow \mathbb{P}T(X)$ with a commuting diagram

$$\begin{array}{ccc} \text{Univ}_{\mathcal{K}} & \xrightarrow{\tau} & \mathbb{P}T(X) \\ \mu_{\mathcal{K}} \downarrow & & \downarrow \pi \\ X & = & X \end{array}$$

such that the tangent morphism $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ is just the fiber of this diagram at a general point $x \in X$. In terms of τ , we can summarize Theorem 3.4, Theorem 3.15 and Theorem 3.17 as follows.

Theorem 3.18. *Let \mathcal{K} be an unbreakable uniruling on a projective manifold X and let $\tau : \text{Univ}_{\mathcal{K}} \dashrightarrow \mathbb{P}T(X)$ be the tangent map. Then there exists a Zariski open subset $X_o \subset X$ satisfying the following properties.*

- (i) *The dense open subset $\text{Univ}_{\mathcal{K}}^o := \mu_{\mathcal{K}}^{-1}(X_o)$ of $\text{Univ}_{\mathcal{K}}$ is a smooth quasi-projective variety.*
- (ii) *The morphism $\mu_{\mathcal{K}}$ restricted to $\text{Univ}_{\mathcal{K}}^o$ is a submersion over X_o .*
- (iii) *The restriction of τ gives a morphism $\tau^o : \text{Univ}_{\mathcal{K}}^o \rightarrow \mathbb{P}T(X_o)$ which is normalization of its image.*
- (iv) *The two vector subbundles $T^{\mu\kappa}$ and $T^{\rho\kappa}$ of $T(\text{Univ}_{\mathcal{K}}^o)$ are transversal.*

In fact, (i) and (ii) follow from Theorem 3.4, (iii) follows from Theorem 3.17 and finally (iv) follows from Theorem 3.14.

In this section, we have collected some of the key results on minimal rational curves that are needed for the next section. There are many other results on minimal rational curves which we have omitted, for which we refer the reader to [28] and the survey papers cited in Section 1.

4. FROM MORI TO CARTAN: VMRT-STRUCTURES

In Section 3, we have seen that starting from a uniruled projective manifold X by choosing an unbreakable uniruling \mathcal{K} and a general point $x \in X$, we obtain \mathcal{K}_x , a finite union of projective manifolds. Since $\dim \mathcal{K}_x < \dim X$ by Theorem 3.8, the variety \mathcal{K}_x is likely to be simpler than X . This opens up the possibility of using \mathcal{K}_x to study the geometry of X . Indeed, at least for unirulings by lines, the variety \mathcal{K}_x has been used in this way in many classical constructions. Now Theorem 3.15 and Theorem 3.17 show a bigger advantage of this \mathcal{K}_x : it is provided with a morphism $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ which is almost an embedding, a normalization of its image. That is, starting from the intrinsic information of an unbreakable uniruling, we obtain a natural projective subvariety $\text{Im}(\tau_x) \subset \mathbb{P}T_x(X)$. The extrinsic projective geometry of $\text{Im}(\tau_x)$ yields intrinsic information on X and \mathcal{K} . Since τ_x is just the normalization of its image, it seems more advantageous to look at $\text{Im}(\tau_x)$ rather than \mathcal{K}_x . This motivates the following definition.

Definition 4.1. In the above setting, the image $\text{Im}(\tau_x)$ is denoted by $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ and called the *variety of minimal rational tangents* (abbr. VMRT) at x associated to \mathcal{K} .

This shift of attention from \mathcal{K}_x to \mathcal{C}_x connects Mori geometry to Cartan geometry:

Definition 4.2. In the setting of Theorem 3.18, put $\tilde{\mathcal{C}} := \text{Univ}_{\mathcal{K}}^o$ and $\mathcal{C} := \tau(\text{Univ}_{\mathcal{K}}^o)$. Then $\nu = \tau^o : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is a normalization morphism. Putting $\mathcal{T} := T^{\rho\kappa} \oplus T^{\mu\kappa}$ on $\tilde{\mathcal{C}}$, Theorem 3.18 (iv) says that \mathcal{T} is a vector subbundle of $T(\tilde{\mathcal{C}})$. It satisfies the condition (3)

of Definition 2.2, so $\mathcal{C} \subset \mathbb{P}T(X_o)$ is a cone structure on the complex manifold X_o . This cone structure is called the *VMRT-structure* of the unbreakable uniruling \mathcal{K} . Moreover $T^{\rho\kappa}$ gives a connection \mathcal{F} on this cone structure, called the *tautological connection* on \mathcal{C} .

The following is proved in [14] and [21].

Theorem 4.3. *The tautological connection \mathcal{F} on the VMRT-structure in Definition 4.2 is a \mathcal{P} -splitting characteristic connection.*

It follows that a choice of an unbreakable uniruling on a uniruled projective manifold gives rise to a cone structure with a natural \mathcal{P} -splitting characteristic connection: the VMRT-structure with the tautological connection. This provides diverse examples of such cone structures. How much diversity? We will see shortly that this transition from algebraic geometry to differential geometry

$$(X, \mathcal{K}) \Rightarrow (\mathcal{C} \subset \mathbb{P}T(X_o), \mathcal{F})$$

is injective, under some topological assumptions. Thus one may say that such cone structures are almost as diverse as all uniruled projective manifolds and all unbreakable unirulings on them. Referring the reader to [9] and [19] for many interesting examples of VMRT-structures, let us just note three salient features of this diversity.

Firstly, for any irreducible smooth projective variety $Z \subset \mathbb{P}V$, there exists a VMRT-structure which is Z -isotrivial and locally flat (Example 1.7 in [12]). In fact, writing $W = V \oplus \mathbb{C}$, we can regard $Z \subset \mathbb{P}V \subset \mathbb{P}W$ as a smooth subvariety of $\mathbb{P}W$ contained in the hyperplane $\mathbb{P}V$. Then the blow-up $X := \text{Bl}_Z(\mathbb{P}W)$ has an unbreakable uniruling whose general members are proper transforms of lines in $\mathbb{P}W$ intersecting Z at one point. Then for $x \in X$ over a point on $\mathbb{P}W \setminus \mathbb{P}V$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to $Z \subset \mathbb{P}V$ as projective varieties. It is easy to see that this VMRT-structure is locally flat.

Secondly, there are isotrivial VMRT-structures which are not locally flat. A simple example of this type is provided by a homogeneous contact manifold X with second Betti number 1, different from projective space. This is a homogeneous projective manifold $X = G/P$ equipped with a G -invariant holomorphic contact structure $\mathcal{D} \subset T(X)$, i.e., a Pfaffian system of corank 1 which is maximally non-integrable. It has a uniruling by lines, the VMRT-structure $\mathcal{C} \subset \mathbb{P}T(X)$ of which is G -invariant and hence it is isotrivial. Furthermore, at each $x \in X$, the VMRT \mathcal{C}_x is contained in $\mathbb{P}\mathcal{D}_x$ and, in fact, spans $\mathbb{P}\mathcal{D}_x$ (see [8]). This implies that \mathcal{C} is not locally flat, because if it were, then \mathcal{D} would be integrable.

Thirdly, there are many examples of non-isotrivial VMRT-structures. Actually, most of the VMRT-structures on projective manifolds with second Betti number 1 are expected to be non-isotrivial. A most transparent example is given by the moduli variety $\text{SU}_C(r, d)$ of stable bundles on a curve C of genus greater than 3 with rank r and fixed determinant of degree r . When $(r, d) = 1$, this is a projective manifold with second Betti number 1. There is a minimal uniruling given by ‘Hecke curves’, certain families of stable bundles on C arising from a geometric version of the Hecke correspondence introduced in [38]. Its VMRT’s are iterated projective bundles on C constructed from the universal bundle on

$C \times \mathrm{SU}_C(r, d)$, which shows that the VMRT-structure is not isotrivial (see [10] and [23] for details).

These three points illustrate the diversity of VMRT-structures. But even on these specific points, our understanding of this diversity is very limited.

Regarding the examples of locally flat VMRT-structures, our construction does not work if Z is reducible. In fact, if \mathcal{K}_x is reducible, there is a global constraint on the VMRT-structure coming from the irreducibility of \mathcal{K} . For example, we cannot have \mathcal{C}_x consisting of two components only one of which is linear. In fact, if this were to happen, then collecting the linear components for general $x \in X$, we would get an irreducible component of \mathcal{C} different from \mathcal{C} , a contradiction to the irreducibility of \mathcal{C} . Also the construction does not work if Z is singular with smooth normalization. Which singular varieties can be realized as the VMRT of an unbreakable uniruling is a completely open question.

As for the examples of non-isotrivial VMRT-structures, although we expect that ‘most’ VMRT-structures are non-isotrivial, this has been verified only in a small number of cases because checking the non-isotriviality can be tricky even when X is a simple well-known variety. We do not have a general method for doing this. An approach to prove that a certain type of VMRT-structures are not isotrivial has been proposed in [12], but this has not been implemented in concrete examples. So far the non-isotriviality is known only for a handful of examples, and only by explicit direct description of the VMRT’s. Besides $\mathrm{SU}_C(r, d)$ explained above, there are only two more types of examples we know of. In [29], Landsberg and Robles show that the uniruling by lines on a general smooth hypersurface of degree d , $3 \leq d \leq n$, in \mathbb{P}^{n+1} has non-isotrivial VMRT-structure. Another case is in [16] which shows the non-isotriviality of the VMRT-structure for the uniruling by lines on a double cover of \mathbb{P}^n branched along a general smooth hypersurface of degree $2m$, $2 \leq m \leq n - 1$.

Now that we have many cone structures with \mathcal{P} -splitting characteristic connections arising from unbreakable unirulings on projective manifolds, it is natural to study the local equivalence problem for these structures in the sense of Cartan, which we can formulate as follows.

Definition 4.4. Let X^1 (resp. X^2) be a uniruled projective manifold with an unbreakable uniruling \mathcal{K}^1 (resp. \mathcal{K}^2). Let $\mathcal{C}^1 \subset \mathbb{P}T(X_o^1)$ (resp. $\mathcal{C}^2 \subset \mathbb{P}T(X_o^2)$) be the corresponding VMRT-structure with the tautological connection \mathcal{F}^1 (resp. \mathcal{F}^2). Suppose that there exists a connected Euclidean open subset $U^1 \subset X_o^1$ (resp. $U^2 \subset X_o^2$) and a biholomorphic map $\varphi : U^1 \rightarrow U^2$ such that $\mathrm{Pd}\varphi : \mathbb{P}T(U^1) \rightarrow \mathbb{P}T(U^2)$ sends $\mathcal{C}^1|_{U^1}$ to $\mathcal{C}^2|_{U^2}$ and $\mathcal{F}^1|_{U^1}$ to $\mathcal{F}^2|_{U^2}$. Then we say that the VMRT-structures \mathcal{C}^1 and \mathcal{C}^2 are *locally equivalent* and $\varphi : U^1 \rightarrow U^2$ is a *local equivalence map* of the two VMRT-structures.

This is of course a natural definition from the viewpoint of Cartan geometry. But does it have significant implications in algebraic geometry? The following result from [20] shows that this is indeed so, under some topological assumptions.

Theorem 4.5. *In the setting of Definition 4.4, assume that $b_2(X^1) = 1$ and $\dim \mathcal{K}_x^1 > 0$ for a general $x \in X_1$, or equivalently, $\deg_{K_{X^1}^{-1}}(\mathcal{K}^1) \geq 3$. Then an equivalence map φ can be extended to a rational map $\Phi : X^1 \dashrightarrow X^2$, i.e., $\varphi = \Phi|_{U^1}$. If furthermore, $b_2(X^2) = 1$, then Φ is biregular.*

In other words, within the class of uniruled projective manifolds of second Betti number 1 and unirulings of anti-canonical degree at least 3, the correspondence

$$(X, \mathcal{K}) \Rightarrow (\mathcal{C} \subset \mathbb{P}T(X_o), \mathcal{F})$$

is injective. The requirement in Theorem 4.5 that the projective manifolds have second Betti number 1 is necessary. Indeed, it is easy to construct examples of X^1 with $b_2(X^1) > 1$ where an analogue of Theorem 4.5 fails. However, this condition is not a serious handicap. In fact, projective manifolds with second Betti number 1 form a large class of projective varieties where general structure theories of higher dimensional algebraic geometry, like the minimal model program, give little direct information. That VMRT-theory is effective for uniruled projective manifolds with second Betti number 1 means that it could complement these general structure theories.

The proof of Theorem 4.5 is by analytic continuation of the map φ along members of \mathcal{K}^1 . This analytic continuation corresponds to the construction of the developing map in Cartan geometry. Here X^2 is regarded as the model of the geometric structure and the map φ is developed to Φ . The topological condition $b_2(X^1) = 1$ is used to guarantee that the analytic continuation can be carried out to cover the whole of X^1 . The potential multi-valuedness of the analytic continuation is taken care of by the condition $\dim \mathcal{K}_x^1 > 0$.

From Definition 4.4, the condition required for the map φ in Theorem 4.5 is that $\mathbb{P}d\varphi$ preserves the cone structure and also the characteristic connection. These are differential geometric conditions. To be useful in algebraic geometry, we need a way to find algebro-geometric conditions to guarantee them. In most interesting cases, the requirement of preserving the connection can be replaced by algebro-geometric condition in the following form:

Theorem 4.6. *Let X^1 (resp. X^2) be a uniruled projective manifold with $b_2(X^1) = 1$ (resp. $b_2(X^2) = 1$). Let \mathcal{K}^1 (resp. \mathcal{K}^2) be an unbreakable uniruling on X^1 (resp. X^2). Let $\mathcal{C}^1 \subset \mathbb{P}T(X_o^1)$ (resp. $\mathcal{C}^2 \subset \mathbb{P}T(X_o^2)$) be the corresponding VMRT-structure. Suppose that there exists a connected Euclidean open subset $U^1 \subset X_o^1$ (resp. $U^2 \subset X_o^2$) and a biholomorphic map $\varphi : U^1 \rightarrow U^2$ such that $\mathbb{P}d\varphi : \mathbb{P}T(U^1) \rightarrow \mathbb{P}T(U^2)$ sends $\mathcal{C}^1|_{U^1}$ to $\mathcal{C}^2|_{U^2}$. If the VMRT \mathcal{C}_x^1 at a general point $x \in U^1$ has positive dimension and nondegenerate Gauss map, then there exists a biregular morphism $\Phi : X^1 \rightarrow X^2$ such that $\Phi|_{U^1} = \varphi$.*

This is a simple combination of Theorem 2.17 and Theorem 4.5. Note that one Cartanian condition in Theorem 4.5, i.e., that φ preserves the connection \mathcal{F} , has been replaced by an algebro-geometric condition on the fiber \mathcal{C}_x , the nondegeneracy of Gauss map. The latter condition holds for the VMRT of a large class of uniruled projective manifolds and unbreakable unirulings, so we may concentrate on VMRT-structures satisfying Theorem

4.6. Thus our main problem becomes finding algebro-geometric conditions for the local equivalence of the cone structures. In this regard, the most central problem connecting Mori geometry and Cartan geometry is the following:

Problem 4.7. Let X^1 (resp. X^2) be a uniruled projective manifold with an unbreakable uniruling \mathcal{K}^1 (resp. \mathcal{K}^2). Let $\mathcal{C}^1 \subset \mathbb{P}T(X_o^1)$ (resp. $\mathcal{C}^2 \subset \mathbb{P}T(X_o^2)$) be the corresponding VMRT-structure with the tautological connection \mathcal{F}^1 (resp. \mathcal{F}^2). Suppose that there exists a connected Euclidean open subset $U^1 \subset X_o^1$ (resp. $U^2 \subset X_o^2$) and a biholomorphic map $\varphi : U^1 \rightarrow U^2$ such that $\mathcal{C}^1|_{U^1}$ and $\mathcal{C}^2|_{U^2}$ are isomorphic as families of projective varieties. More precisely, suppose that we have a biholomorphic map $\psi : \mathbb{P}T(U^1) \rightarrow \mathbb{P}T(U^2)$ satisfying $\psi(\mathcal{C}^1|_{U^1}) = \mathcal{C}^2|_{U^2}$ with a commuting diagram

$$\begin{array}{ccc} \mathbb{P}T(U^1) & \xrightarrow{\psi} & \mathbb{P}T(U^2) \\ \downarrow & & \downarrow \\ U^1 & \xrightarrow{\varphi} & U^2 \end{array} .$$

Does this imply that the two cone structures $\mathcal{C}^1|_{U^1}$ and $\mathcal{C}^2|_{U^2}$ are locally equivalent? In other words, can we choose φ such that $\psi = \mathbb{P}d\varphi$?

What makes Problem 4.7 exciting and challenging is that it needs insights and techniques for the fusion of the algebraic geometry of the projective variety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ and the differential geometry of the cone structure \mathcal{C} via the connection \mathcal{F} . At present, no plausible uniform method to handle Problem 4.7 seems conceivable and, depending on the type of the family of varieties \mathcal{C}_x 's, different tools are required. A practicable approach is to try the cases where the projective geometry of the family \mathcal{C}_x 's is simple and the differential geometric machinery of the cone structure is available. A most reasonable candidate is the case when the VMRT-structure is Z -isotrivial for a projective variety $Z \subset \mathbb{P}V$ of simple type.

Even for Z -isotrivial cases, Problem 4.7 is highly challenging. The answer is not always affirmative. There are examples of two uniruled projective manifolds with Z -isotrivial VMRT-structures for the same smooth irreducible $Z \subset \mathbb{P}V$, which are not locally equivalent. Since we have seen that a locally flat Z -isotrivial VMRT-structure exists for any given $Z \subset \mathbb{P}V$, it suffices to give examples of Z -isotrivial VMRT-structures that are not locally flat. We have already mentioned that homogeneous contact manifolds provide such examples where $Z \subset \mathbb{P}V$ is contained in a hyperplane in $\mathbb{P}V$. There are also examples where Z spans $\mathbb{P}V$. A simple example is provided by symplectic Grassmannians. Given a vector space W with a symplectic form ω , let us denote by $\text{Gr}_\omega(k, W)$ the set of k -dimensional subspaces of W isotropic with respect to ω . This projective manifold $\text{Gr}_\omega(k, W)$ has a uniruling by lines. The associated VMRT-structure is Z -isotrivial for some Z because $\text{Gr}_\omega(k, W)$ is a homogeneous space. It turns out that $Z \subset \mathbb{P}V$ is nondegenerate. If $2k < \dim W$, the automorphism group $\text{Aut}(\widehat{Z})$ acts on Z with two orbits and the unique closed orbit $Z' \subset Z$ is degenerate. The corresponding subvariety $\mathcal{C}' \subset \mathcal{C}$ of the VMRT-structure spans a Pfaffian system $\mathcal{D} \subset T(\text{Gr}_\omega(k, W))$ which is not integrable. This non-integrability implies that the VMRT-structure is not locally flat (see [22] for details).

This example shows that the answer to Problem 4.7 for Z -isotrivial cases would depend on the variety Z . This raises the following purely local problem in Cartan geometry:

Problem 4.8. Find algebro-geometric conditions on a smooth irreducible $Z \subset \mathbb{P}V$ which guarantee that every Z -isotrivial cone structure with a \mathcal{P} -splitting characteristic connection is locally flat.

As we have already mentioned in Section 2, there are two types of Z where the answer is known. When Z is a linear subspace of $\mathbb{P}V$, we have seen that the corresponding Pfaffian system is involutive, thus the cone structure is locally flat. Another case is when $Z \subset \mathbb{P}V$ is a smooth hypersurface of degree ≥ 3 by Theorem 2.19 together with the comment after Theorem 2.21. At the moment, these are the only cases where Problem 4.8 has been answered, even though it is reasonable to expect that it is locally flat for most choices of $Z \subset \mathbb{P}V$.

One important case of Z -isotrivial VMRT-structures where an affirmative answer is obtained for Problem 4.7 is when Z is the VMRT of an irreducible Hermitian symmetric space. This is not done through the local approach of Problem 4.8. Instead, a mixture of local and global methods has settled the question successfully. As mentioned in Section 2, differential geometric machinery of such structures has been developed by differential geometers and there are natural curvature tensors whose vanishing implies the local flatness. In [34], Mok shows that if a VMRT-structure is Z -isotrivial where Z is the VMRT of an irreducible Hermitian symmetric space, then the Z -isotrivial cone structure can be extended to a neighborhood of a general member of the unbreakable uniruling. Then the unbending property of the rational curve can be used to show the vanishing of the curvature tensors. This way, Mok proves

Theorem 4.9. *Let X be a uniruled projective manifold with second Betti number 1. Suppose there exists an unbreakable uniruling whose VMRT-structure is Z -isotrivial where Z is the VMRT of an irreducible Hermitian symmetric space G/P . Then X is biregular to G/P .*

This resolves Problem 1.3 for irreducible Hermitian symmetric spaces. Mok's method has been extended to any rational homogeneous space G/P where P is a maximal parabolic subgroup of a complex simple Lie group G , associated to a long root of G in [6] and [34]. Whether this remains true when P is associated to a short root of G , such as symplectic Grassmannians, is a most tantalizing problem in this direction. Also, it would be desirable to recover Theorem 4.9 by answering Problem 4.8 for the corresponding Z .

Combining Theorem 4.9 with the affirmative answer to Problem 4.8 when Z is a smooth hypersurface of degree ≥ 3 , we have the following result in [14].

Theorem 4.10. *Let X be a uniruled projective manifold of dimension ≥ 4 with second Betti number 1. Assume that there is an unbreakable uniruling such that the VMRT at a general point is a smooth hypersurface. Then X is a quadric hypersurface.*

These few examples are all the results we currently have for Problem 4.7 for Z -isotrivial cases. They correspond to the simplest type of projective varieties, hypersurfaces and some

homogeneous varieties. Any other projective variety $Z \subset \mathbb{P}V$ seems to be a new challenge. Even less -essentially no concrete result- is known for non-isotrivial cases of Problem 4.7.

There is a natural extension of the equivalence problem for VMRT-structures to the setting of submanifold geometry. In the investigation of any type of geometric structures, it is natural to study submanifolds inheriting such structures. From this perspective, among submanifolds in a uniruled projective manifold X , of particular importance are uniruled submanifolds with VMRT-structures which are compatible with the VMRT-structure of the ambient manifold. The study of such submanifolds from an intrinsic viewpoint, which has been initiated in [35], is based on the fact that the analytic continuation Theorem 4.5 can be easily extended to submanifold geometry in the following sense.

Theorem 4.11. *Let X^1 (resp. X^2) be a uniruled projective manifold with an unbreakable uniruling \mathcal{K}^1 (resp. \mathcal{K}^2). Let $\mathcal{C}^1 \subset \mathbb{P}T(X_o^1)$ (resp. $\mathcal{C}^2 \subset \mathbb{P}T(X_o^2)$) be the corresponding VMRT-structure with the tautological connection \mathcal{F}^1 (resp. \mathcal{F}^2). Suppose that there exists a connected Euclidean open subset $U^1 \subset X_o^1$ (resp. $U^2 \subset X_o^2$) and an embedding $\varphi : U^1 \rightarrow U^2$ such that $\mathbb{P}d\varphi : \mathbb{P}T(U^1) \rightarrow \mathbb{P}T(U^2)$ sends $\mathcal{C}^1|_{U^1}$ into $\mathcal{C}^2|_{U^2}$ and \mathcal{F}^1 into \mathcal{F}^2 . Assume that $b_2(X^1) = 1$ and $\dim \mathcal{K}_x^1 > 0$ for a general $x \in X_1$. Then φ can be extended to a rational map $\Phi : X^1 \dashrightarrow X^2$, i.e., $\varphi = \Phi|_{U^1}$.*

The first task is to replace the preservation of connections by algebro-geometric conditions, just as Theorem 4.6 is an improvement over Theorem 4.5. This is done in [7] and the following analogue of Theorem 4.6 in submanifold geometry is obtained.

Theorem 4.12. *Let X^1 (resp. X^2) be a uniruled projective manifold with a VMRT-structure $\mathcal{C}^1 \subset \mathbb{P}T(X_o^1)$ (resp. $\mathcal{C}^2 \subset \mathbb{P}T(X_o^2)$). Assume that $b_2(X^1) = 1$ and $\dim \mathcal{C}_x^1 > 0$ for $x \in X_o^1$. Suppose that there exists a connected Euclidean open subset $U^1 \subset X_o^1$ (resp. $U^2 \subset X_o^2$) and an embedding $\varphi : U^1 \rightarrow U^2$ satisfying the following two conditions.*

- (i) $\mathbb{P}d\varphi(\mathcal{C}^1|_{U^1}) \subset \mathcal{C}^2|_{\varphi(U^1)}$ and
- (ii) *for a general point $z \in U^1$ with $x = \varphi(z)$ and a general smooth point $\beta \in \mathcal{C}_z^1$, the image $\alpha := \mathbb{P}d\varphi(\beta)$ is a smooth point of \mathcal{C}_x^2 and*

$$\{v \in T_\alpha(\mathcal{C}_x^2), \Pi_{\mathcal{C}_x^2, \alpha}(v, u) = 0 \text{ for all } u \in d(\mathbb{P}d\varphi)(T_\beta(\mathcal{C}_z^1))\} = 0.$$

Then there exists a rational map $\Phi : X^1 \dashrightarrow X^2$ such that $\Phi|_{U^1} = \varphi$.

To be precise, the statement in [7] is slightly weaker, although their argument essentially proves Theorem 4.12. The full statement of Theorem 4.12 is given in [15] with a simplified proof.

The condition (ii) in Theorem 4.12 is a natural extension of the Gauss map condition in Theorem 4.6. Note that the Cartanian condition in Theorem 4.11 on the characteristic connection is replaced by the algebro-geometric condition (ii).

Theorem 4.11 goes beyond Theorem 4.6 even when $\dim X^1 = \dim X^2$ because $\dim \mathcal{K}^1$ can be strictly smaller than $\dim \mathcal{K}^2$. For example, it can describe a finite morphism $X^1 \rightarrow X^2$ which sends members of \mathcal{K}^1 to members of \mathcal{K}^2 .

One immediate question is whether the map Φ can be extended to a morphism $\Phi : X^1 \rightarrow X^2$ when $b_2(X^2) = 1$, as in Theorem 4.6. In [7], an affirmative answer is given when X^1 is modeled on a special class of Schubert submanifolds in rational homogeneous spaces $X^2 = G/P$.

At present, submanifold theory in the interaction of Mori geometry and Cartan geometry is in an incipient stage. Many natural questions can be raised, but the most fundamental one is an analogue of Problem 4.7 in the submanifold setting. This would lead to a deeper aspect of Cartan geometry. Interesting applications to algebraic geometry are yet to come.

In conclusion, VMRT-structures on uniruled projective manifolds provide us with a large number of examples in great diversity of cone structures admitting \mathcal{P} -splitting characteristic connections. Cartan geometry of these structures has been understood for only a few special cases and a wide range of examples and problems remain to be explored. The small number of results we have seen so far have already found interesting applications in algebraic geometry. Further development will undoubtedly bring more exciting applications. This will be a fertile ground for interactions between differential geometry and algebraic geometry.

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